

Descriptive Dynamics

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The purpose of the following informal lectures is to give a brief introduction to *descriptive dynamics*, which I understand here to be the *descriptive theory of Polish group actions*. I will concentrate on the foundations, and hopefully at a level accessible to anyone with a basic knowledge of descriptive set theory. I will illustrate some of the main methods used in this area, including Baire category arguments and various implementations of the “changing the topology” technique. A general reference for the results discussed in this paper is Becker-Kechris [1996].

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Lecture I

A Polish Groups

Classically in various branches of dynamics one studies actions of the group of integers (\mathbb{Z}), reals (\mathbb{R}), Lie groups, or even more generally (second countable) locally compact groups. We expand here this scope by considering the more comprehensive class of *Polish groups*, which seems to be the widest class of well-behaved (for our purposes) groups and which includes practically every type of topological group we are interested in.

Definition. A *Polish group* is a topological group whose topology is Polish (i.e., separable, completely metrizable).

Here are some examples of Polish groups (for which more details are given in Becker-Kechris [1996] and Kechris [1995]).

Examples.

1. (Second countable) locally compact groups;
2. Separable Banach spaces (under $+$);
3. Various groups of symmetries of mathematical objects. Here are some concrete examples:

- (a) S_∞ , the infinite symmetric group of all permutations of \mathbb{N} , with the topology of pointwise convergence.
- (b) Consider an arbitrary countable structure

$$\mathcal{A} = \langle A, f, g, \dots, R, S, \dots \rangle$$

consisting of a countable set A equipped with certain operations f, g, \dots and certain relations R, S, \dots (each of varying numbers of arguments). Typical examples include: groups $\langle G, \cdot \rangle$, fields $\langle F, +, \cdot \rangle$, ordered groups $\langle G, \cdot, \leq \rangle$, graphs $\langle V, E \rangle$, etc. Let $\text{Aut}(\mathcal{A})$ be the automorphism group of \mathcal{A} . This is a Polish group, again with the topology of pointwise convergence. When A is infinite we can take, without loss of generality, $A = \mathbb{N}$ and in this case $\text{Aut}(\mathcal{A})$ is a closed subgroup of S_∞ . Conversely, every closed subgroup of S_∞ is of that form for an appropriate \mathcal{A} (with $A = \mathbb{N}$).

- (c) $U(H)$, the unitary group of a separable Hilbert space H , with the weak (or equivalently strong) topology.
- (d) $H(X)$, the homeomorphism group of a compact metrizable space X , with the uniform topology.
- (e) $\text{Iso}(X, d)$, the isometry group of a complete separable metric space (X, d) , with the pointwise convergence topology.
- (f) $\text{Aut}(X, \mu)$ (resp. $\text{Aut}^*(X, \mu)$), the group of measure preserving (resp. nonsingular, i.e., null set preserving) transformations of a standard probability measure space (X, μ) (X is a *standard Borel space*, i.e., a Polish space equipped with its σ -algebra of Borel sets and μ is a Borel probability measure on X). These can be viewed, by the usual association of a unitary operator to each transformation, as closed subgroups of $U(L^2(X, \mu))$, up to (topological group) isomorphism.

We often consider subclasses of Polish groups with various additional nice properties. These can have algebraic flavor, as, for example, the classes of abelian, nilpotent, and solvable groups, or topological flavor, as, for example, the classes of locally compact, admitting invariant metric (which must necessarily be complete), or admitting *complete* left-invariant metric Polish groups. See Becker [1998] for a more complete exposition. Here are the inclusions among these classes: abelian \subsetneq nilpotent \subsetneq solvable, abelian \subsetneq inv. metric, l.c. \cup solvable \cup inv. metric \subsetneq l.inv.complete metric \subsetneq Polish.

Another interesting class of Polish groups consists of those which have a countable local basis at 1 consisting of open subgroups. These are exactly the

closed subgroups of S_∞ or equivalently the automorphism groups of countable structures (up to isomorphism). See Becker-Kechris [1996].

Finally, there is a universal Polish group:

Theorem (Uspenskii [1986]). *The group $H([0, 1]^{\mathbb{N}})$ of homeomorphisms of the Hilbert cube is a universal Polish group, i.e., every Polish group is isomorphic to a closed subgroup of it.*

B Polish and Borel G -spaces

Definition. Let G be a Polish group. A *Polish G -space* is a Polish space X together with a continuous action of G on X . A *Borel G -space* is a standard Borel space X together with a Borel action of G on X .

There are two basic facts concerning Polish and Borel G -spaces.

Theorem (Effros [1965]; see also Becker-Kechris [1996]). *Let G be a Polish group and X a Polish G -space. For each $x \in X$, the following are equivalent, where $G_x = \{g : g \cdot x = x\}$ is the stabilizer of x ;*

- (i) $gG_x \mapsto g \cdot x$ is a homeomorphism of the Polish space G/G_x onto $G \cdot x$ (or equivalently the map $g \mapsto g \cdot x$ is open from G onto $G \cdot x$);
- (ii) $G \cdot x$ is not meager in its relative topology;
- (iii) $G \cdot x$ is G_δ in X .

Corollary. *If $G \cdot x$ is not meager (in X), then it is G_δ in X .*

Theorem (D. Miller [1977]; see also Kechris [1995]). *Let G be a Polish group and X a Borel G -space. Then G_x is closed and $G \cdot x$ is Borel in X .*

However, in general, the orbit equivalence relation

$$xE_G^X y \Leftrightarrow xE_G y \Leftrightarrow \exists g(g \cdot x = y)$$

is (analytic but) not Borel.

Given a closed subgroup $G \subseteq H$ of a Polish group H , there is a canonical "minimal" way to extend a given G -action to an H -action, called the *induced action*. This construction is quite useful in showing that various properties of Polish groups with respect to their actions are hereditary, i.e., are inherited by their closed subgroups.

Theorem (Mackey [1966]; see also Becker-Kechris [1996]). *Let G, H be Polish groups with G a closed subgroup of H . Let X be a Borel G -space. There is a unique, up to Borel H -isomorphism, Borel H -space \tilde{X} and Borel G -injection*

$i: X \rightarrow \tilde{X}$ such that for every Borel H -space Y and Borel G -map $f: X \rightarrow Y$, there is a unique Borel H -map $\tilde{f}: \tilde{X} \rightarrow Y$ so that $\tilde{f} \circ i = f$. (If X is a G -space and Y is an H -space, a map f is a G -map if $f(g \cdot x) = g \cdot f(x)$, for $g \in G, x \in X$.)

The space \tilde{X} is denoted by $H \times_G X$ and can be realized as follows:

$\tilde{X} = (H \times X)/G :=$ the orbit space of the action of G on $H \times X$ given by $g \cdot (h, x) = (g \cdot h, g \cdot x)$, with the quotient Borel structure; the action of H on \tilde{X} is given by

$$h \cdot [h', x] = [h'h^{-1}, x]$$

and

$$i(x) = [1, x].$$

Moreover, identifying X with $i(X)$, note that every H -orbit of Y contains a unique G -orbit of X , a fact which is often quite useful.

One also has the analog of the preceding theorem in the topological context.

Theorem (Hjorth; see Becker-Kechris [1996]). *Let G, H be Polish groups with G a closed subgroup of H . Let X be a Polish G -space. There is a unique, up to H -homeomorphism, Polish H -space \tilde{X} and G -homeomorphism $i: X \rightarrow \tilde{X}$ such that for every Polish H -space Y and continuous G -map $f: X \rightarrow Y$, there is a unique continuous H -map $\tilde{f}: \tilde{X} \rightarrow Y$ so that $\tilde{f} \circ i = f$.*

The space $\tilde{X} = H \times_G X$ is defined exactly as before with the quotient topology now. It also turns out that $i(X)$ is closed in \tilde{X} .

C Universal G -spaces

Definition. Let G be a Polish group. A Borel G -space \mathcal{U} is *universal* if every Borel G -space can be Borel G -embedded into \mathcal{U} .

It is easy to see that such a space is unique up to Borel G -isomorphism, if it exists. We will denote it by \mathcal{U}_G .

Theorem (Mackey [1962], Varadarajan [1963]). *If G is Polish locally compact, then \mathcal{U}_G exists and can be realized as a compact Polish G -space (i.e., the space acted upon is compact metrizable).*

This was extended recently to arbitrary Polish groups.

Theorem (Becker-Kechris [1996]). *If G is Polish, then \mathcal{U}_G exists and can be realized as a compact G -space.*

There are several realizations of the universal G -space \mathcal{U}_G which are useful in different circumstances.

Realization 1. Let G be a Polish group and $\mathcal{F}(G)$ the standard Borel space of closed subsets of G with the Effros Borel structure. Let G act on $\mathcal{F}(G)$ by left-translation:

$$g \cdot F = gF.$$

Then the infinite product G -space

$$\mathcal{U}_G^1 = \mathcal{F}(G)^{\mathbb{N}}$$

is universal; see Becker-Kechris [1996].

Proof. In fact we will show that any Borel action of G on a separable metrizable space Borel G -embeds into \mathcal{U}_G^1 . Fix a countable open basis $\{U_n\}$ for X . Clearly it separates points. For $A \subseteq G$, let

$$E(A) = \{g \in G : \text{For every open nbhd } V \text{ of } g, V \cap A \text{ is not meager}\}.$$

Then $E(A)$ is closed and, if A has the Baire property, $A \Delta E(A)$ is meager. Put

$$\begin{aligned} \pi &= (\pi_n) : X \rightarrow \mathcal{F}(G)^{\mathbb{N}}, \\ \pi_n(x) &= E(\{g : g \cdot x \in U_n\})^{-1}. \end{aligned}$$

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Notice that if G is locally compact Polish and we give $\mathcal{F}(G)$ the Fell topology, then \mathcal{U}_G^1 becomes a compact Polish G -space.

Realization 2. Let G be a Polish group and d a left-invariant compatible metric, $d < 1$. Let

$$\mathcal{L}_d(G) \equiv \mathcal{L}(G) = \{f : G \rightarrow [0, 1] : |f(g) - f(h)| \leq d(g, h)\}$$

with the topology it inherits as a subset of $[0, 1]^G$ (with the product topology), so it is compact metrizable. Let G act on $\mathcal{L}(G)$ by left-shift

$$g \cdot f(h) = f(g^{-1}h).$$

Then $\mathcal{L}(G)$ is a compact Polish G -space and

$$\mathcal{U}_G^2 = \mathcal{L}(G)^{\mathbb{N}}$$

is universal.

Proof. Embed $\mathcal{F}(G)$ into $\mathcal{L}(G)$ by

$$F \mapsto (g \mapsto d(g, F)).$$

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Realization 3 (Gao [1996]). Let G be a Polish group and d a left-invariant compatible metric, $d < 1$. Fix a dense set $\{g_n\}$ in G . It is well-known that the map $\pi : G \rightarrow [0, 1]^{\mathbb{N}}$, given by $\pi(g) = (d(g, g_n))_{n \in \mathbb{N}}$, is a homeomorphism. Denote by \bar{G} the closure of $\pi(G)$ in $[0, 1]^{\mathbb{N}}$. Then one can extend to \bar{G} the left-translation action of G on itself, by defining

$$g \cdot x = \lim_n \pi(g h_n)$$

for $x \in \bar{G}$, $h_n \in G$, $\pi(h_n) \rightarrow x$. It turns out that \bar{G} with this action is a compact Polish G -space.

Consider $\mathcal{K}(\bar{G})$, the hyperspace of all compact subsets of \bar{G} with the Vietoris topology, which is a compact metrizable space. G acts continuously on it by $g \cdot K = \{g \cdot x : x \in K\}$ and so $\mathcal{K}(\bar{G})$ is a compact Polish G -space. Let

$$\mathcal{U}_G^3 = \mathcal{K}(\bar{G})^{\mathbb{N}}.$$

Since the map $F \in \mathcal{F}(G) \mapsto \overline{\pi(F)} \in \mathcal{K}(\bar{G})$ is a Borel G -embedding, it follows that \mathcal{U}_G^3 is universal. —

Realization 4, for Polish locally compact G (Mackey [1962], Varadarajan [1963]). Let G be Polish locally compact, μ_G its Haar measure and put

$$\mathcal{U}_G^4 = B_1(L^\infty(G, \mu_G), w^*),$$

the unit ball of $L^\infty(G, \mu_G)$ with the weak*-topology, and G acting on \mathcal{U}_G^4 by left-shift again. Then \mathcal{U}_G^4 is a compact Polish G -space which is universal.

Proof. Fix a Borel G -space $X \subseteq [0, 1]$. Embed it into \mathcal{U}_G^4 by

$$x \mapsto f_x(g) = g \cdot x.$$

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D Applications

We will now present some applications of the universal space.

1. Actions of S_∞ and model theory

Consider structures of the form $\mathcal{A} = \langle \mathbb{N}, R_1, R_2, \dots \rangle$, with $R_n \subseteq \mathbb{N}^n$ ("hyper-graphs"). Each such structure can be identified with an element of

$$\mathcal{U}_{S_\infty} = \prod_n 2^{\mathbb{N}^n}.$$

S_∞ acts on this space by $g \cdot \langle \mathbb{N}, R_1, R_2, \dots \rangle = \langle \mathbb{N}, R'_1, R'_2, \dots \rangle$, where

$$R'_n(a_1, \dots, a_n) \Leftrightarrow R_n(g(a_1), \dots, g(a_n)).$$

This is called the *logic action*. Clearly $\mathcal{A}, \mathcal{A}' \in \mathcal{U}_{S_\infty}$ belong to the same orbit iff $\mathcal{A} \cong \mathcal{A}'$, i.e., the orbits are simply the isomorphism classes. One can use Realization 1 to show that this is a universal S_∞ -space (see Becker-Kechris [1996]). So if X is an arbitrary Borel S_∞ -space, there is a Borel S_∞ -embedding $f: X \rightarrow \mathcal{U}_{S_\infty}$, and so in particular $f(X)$ is an invariant under \cong Borel subset of \mathcal{U}_{S_∞} . By a theorem of Lopez-Escobar (see, e.g., Kechris [1995]), membership in any Borel invariant subset of \mathcal{U}_{S_∞} can be defined by a countable set of axioms, in an appropriate logical language, called $L_{\omega_1\omega}$, which allows, beyond the usual logical operations, $\wedge, \vee, \neg, \exists, \forall$, countable infinitary conjunctions and disjunctions. (A typical example of such axioms are those for the torsion-free abelian groups.) Therefore, any Borel S_∞ -action is Borel isomorphic to the logic action on the models of a countable theory (in $L_{\omega_1\omega}$) and this establishes a basic connection between S_∞ -actions and model theory, which motivates a lot of work in this area. For more on this subject, see Becker-Kechris [1996].

2. Tarski's Theorem

Suppose G is an arbitrary (abstract) group and X an arbitrary G -space. Given $A, B \subseteq X$ we say that A, B are *equivalent by finite decomposition*, in symbols $A \sim B$, if there are partitions $A = \bigcup_{i=1}^n A_i$, $B = \bigcup_{i=1}^n B_i$ and $g_i \in G$, with $g_i \cdot A_i = B_i$. We say that X is *paradoxical* if $X \sim A \sim B$, with $A \cap B = \emptyset$. A well-known result in the theory of paradoxical decompositions is the following:

Theorem (Tarski; see Wagon [1993]). *For any group G and any G -space X , the following are equivalent:*

- (i) *There is a finitely additive probability measure on X (defined for all subsets of X) which is G -invariant;*
- (ii) *X is not paradoxical.*

The problem has been raised (see, e.g., Wagon [1993]) whether there is an analog of Tarski's theorem for ordinary, countably additive measures. The context is as follows.

Assume (X, \mathcal{S}) is a measurable space and G acts on X preserving \mathcal{S} , i.e., $g \in G, A \in \mathcal{S} \Rightarrow g \cdot A \in \mathcal{S}$. Put for $A, B \in \mathcal{S}$:

$$A \sim_{\infty} B \Leftrightarrow \text{there are partitions } A = \bigcup_{i=1}^{\infty} A_i, B = \bigcup_{i=1}^{\infty} B_i \text{ with} \\ A_i, B_i \in \mathcal{S}, \text{ and } g_i \in G \text{ such that } g_i \cdot A_i = B_i.$$

We say that X is *countably paradoxical* if $X \sim_{\infty} A \sim_{\infty} B$ with $A, B \in \mathcal{S}, A \cap B = \emptyset$. Is it true that X is not countably paradoxical iff there is a (countably additive) probability measure on (X, \mathcal{S}) which is G -invariant?

In this generality it turns out that the answer is negative (see Wagon [1993]), but one can use the existence of universal actions and a theorem of Nadkarni [1990] to show that one gets a positive answer in regular situations.

Theorem (Becker-Kechris [1996]). *Let G be a Polish group and X a Borel G -space. Then the following are equivalent:*

- (i) *There is a Borel probability measure on X which is G -invariant;*
- (ii) *X is not countably paradoxical (for the class of Borel sets).*

Nadkarni [1990] essentially proves this result in the case G is countable. For the general case, the existence of a Polish universal space shows that every Borel G -space is Borel isomorphic to a continuous action of G on a Borel set in a Polish space. One can then apply Nadkarni's theorem to a countable subgroup of G and a straightforward continuity argument to complete the proof.

3. Embedding Polish G -spaces

Now let X be a Polish G -space (G a Polish group). Is it possible to G -embed *topologically* X into a compact Polish G -space Y ? The answer is positive if G is locally compact (see deVries [1978] and Megrelishvili [1989]). Solving a related old problem in the theory of transformation groups, Megrelishvili [1988] showed that the answer is in general negative. Very recently, Scarr [1998] has in fact shown that for a Polish group G , G is locally compact iff every Polish G -space can be G -embedded topologically into a compact Polish G -space. Considering the embedding in the universal space given by

Realization 2, Hjorth and Kechris (see Hjorth [1999a]) proved the following facts:

Theorem (Hjorth-Kechris; see Hjorth [1999a]). *Let G be a Polish group and let \mathcal{U}_G^2 be the universal G -space defined in C, Realization 2. Then \mathcal{U}_G^2 is a compact Polish G -space and for any Polish G -space X there is a G -embedding $\rho : X \rightarrow \mathcal{U}_G^2$ such that*

- (i) $\rho(X)$ is G_δ in \mathcal{U}_G^2 (so Polish).
- (ii) $\rho : X \rightarrow \rho(X)$ is open.
- (iii) $\rho : X \rightarrow \rho(X)$ is Baire class 1.

(So π just misses being a homeomorphism, as expected by Megrelishvili's result.)

Proof. Let ρ be the composition of the embeddings of Realizations 1,2, i.e., $\rho = (\rho_n)$, where

$$\rho_n(x) = (g \mapsto d(g, \{h : h \cdot x \in U_n\}^{-1})),$$

with $\{U_n\}$ an open basis for X (where $d(g, \emptyset) = 1$).

We first check that $\rho : X \rightarrow \rho(X)$ is open. Fix open $U \subseteq X$ in order to show that $\rho(U)$ is open in $\rho(X)$. Let $x \in U$. Then $1 \cdot x \in U$, so find $1 > \varepsilon > 0$ and n such that $x \in U_n$ and for any h with $d(1, h) < \varepsilon$ we have $h^{-1} \cdot U_n \subseteq U$. Consider then the open nbhd $\{\rho(y) : |\rho_n(y)(1) - \rho_n(x)(1)| < \varepsilon\}$ of $\rho(x)$ in $\rho(X)$. It is enough to show it is contained in $\rho(U)$. Fix y in that nbhd. Then $\rho_n(y)(1) = d(1, \{h : h \cdot y \in U_n\}^{-1}) < \varepsilon$ as $\rho_n(x)(1) = 0$ (since $1 \cdot x \in U_n$). So let h be such that $h \cdot y \in U_n$ and $d(1, h^{-1}) < \varepsilon$, so $d(1, h) < \varepsilon$. Then $h^{-1} \cdot U_n \subseteq U$, and so $y \in U$.

For the proof that ρ is Baire class 1 it is enough to check that for each $n, g \in G$, $a \in \mathbb{R}$, the set $\{x : \rho_n(x)(g) < a\}$ is open and the set $\{x : \rho_n(x)(g) > a\}$ is F_σ . The first is straightforward from the definition. For the second, notice that

$$\begin{aligned} \rho_n(x)(g) > a &\Leftrightarrow d(g, \{h : h \cdot x \in U_n\}^{-1}) > a \\ &\Leftrightarrow \exists \text{ rational } b > a (B_g^d(b)^{-1} \cdot x \subseteq U_n^c), \end{aligned}$$

where $B_g^d(b) = \{h : d(h, g) < b\}$ and $U_n^c = X \setminus U_n$.

Finally it remains to show that $\rho(X)$ is G_δ , i.e., Polish in its relative topology. Equivalently if τ^* is the topology on X obtained by transferring the topology of $\rho(X)$ to X via ρ^{-1} we have to show that τ^* is Polish. Since ρ is open, if τ is the topology on X , clearly $\tau \subseteq \tau^*$.

Now a basis for τ^* consists of the sets of the form

$$U_n \cap M_{m_1, g_1, a_1} \cap \dots \cap M_{m_k, g_k, a_k},$$

where

$$M_{m, g, a} = \{x : \rho_m(x)(g) > a\}.$$

(since the sets of the form $\{x : \rho_m(x)(g) < a\}$ are in τ). We can denote this set by $\langle n; m_1, g_1, a_1; \dots; m_k, g_k, a_k \rangle$. To show that (X, τ^*) is Polish, by the Choquet Criterion it is enough to show that II wins the strong Choquet game for this space (see Kechris [1995]).

We describe his strategy below:

If I plays $x_1, \langle n^1; m_1^1, g_1^1, a_1^1; \dots \rangle$, then $\rho_{m_1^1}(x)(g_1^1) > a_j^1$. Fix $\bar{a}_j^1 > a_j^1$ with $\rho_{m_1^1}(x)(g_1^1) > \bar{a}_j^1$ and \bar{n}^1 with $x_1 \in U_{\bar{n}^1} \subseteq \overline{U_{\bar{n}^1}} \subseteq U_{n^1}$ and $\text{diam}(U_{\bar{n}^1}) < \frac{1}{2}$ (in some complete metric for (X, τ)). II responds by playing $\langle \bar{n}^1; m_1^1, g_1^1, \bar{a}_1^1; \dots \rangle$. Next I plays $x_2, \langle n^2; m_1^2, g_1^2, a_1^2; \dots \rangle$. Then define $\langle \bar{n}^2; m_1^2, g_1^2, \bar{a}_1^2; \dots \rangle$ as before, except that $\text{diam}(U_{\bar{n}^2}) < \frac{1}{3}$, and let II play $\langle \bar{n}^2; m_1^2, g_1^2, \bar{a}_1^2; \dots \rangle$, etc. Clearly $x_i \rightarrow x \in \bigcap_k U_{n^k}$ (in τ). Also $x_i \in \{x : \varphi_{m_1^i}(x)(g_1^i) > \bar{a}_1^i\}$ for all i , thus $B_{g_1^1}^d(\bar{a}_1^1)^{-1} \cdot x_i \subseteq U_{m_1^1}^c$, so $B_{g_1^1}^d(\bar{a}_1^1)^{-1} \cdot x \subseteq U_{m_1^1}^c$ and thus $\rho_{m_1^1}(x)(g_1^1) \geq \bar{a}_1^1 > a_1^1$, i.e., $x \in M_{m_1^1, g_1^1, a_1^1}$. Similarly we see that $x \in \langle n^1; m_1^1, g_1^1, a_1^1; \dots \rangle$ and also $x \in \langle n^k; m_1^k, g_1^k, a_1^k; \dots \rangle$ for each k , i.e., x belongs in the intersection of all open sets played by I (and II), so this intersection is non- \emptyset and II won. \dashv

The following was stated as an open problem at the time of the workshop:

Problem. *Is there a universal Polish G -space, i.e., a Polish G -space in which every Polish G -space can be G -embedded topologically?*

The answer was known to be positive for locally compact G (see deVries [1975]). Very recently Hjorth [1999b] solved this, affirmatively again, for *any* Polish group G .

Lecture II

A An Equivariant Version of Kuratowski's Theorem

We will discuss here the "changing the topology" idea, which is quite useful in numerous contexts. We first recall a classical result of Kuratowski, which has many applications in descriptive set theory (see, for example, Kechris [1995]). It shows that, in some sense, Borel sets can be thought as clopen and similarly Borel functions as continuous.

Theorem (Kuratowski).

- (i) Let X be a standard Borel space, $A \subseteq X$ a Borel set. Then there is a Polish topology τ_A , generating the Borel structure of X , with A clopen in (X, τ_A) . If (X, τ) is given as a Polish space, we can also make sure that $\tau_A \supseteq \tau$.
- (ii) If X is a standard Borel space, Y a Polish space and $f : X \rightarrow Y$ is Borel, then there is a Polish topology τ_f generating the Borel structure of X , so that $f : (X, \tau_f) \rightarrow Y$ is continuous. Again if (X, τ) is given as a Polish space, we can take $\tau_f \supseteq \tau$.

This result easily extends to deal simultaneously with countably many A 's or f 's.

We now have the following G -version of this result.

Theorem (Becker-Kechris). Let G be a Polish group. Let X be a Borel G -space, Y a Polish G -space, and $f : X \rightarrow Y$ a Borel G -map. Then there is a Polish topology τ_f on X giving its Borel structure, so that (X, τ_f) is a Polish G -space and $f : (X, \tau_f) \rightarrow Y$ is continuous. If, moreover, (X, τ) is a Polish G -space, then we can take $\tau_f \supseteq \tau$.

Proof. The main tool is the concept of Vaught transform, which plays an important role in descriptive dynamics, and is an application of Baire Category techniques.

Definition. Let G be a Polish group and X a G -space. For $P \subseteq X$, $U \subseteq G$ open non- \emptyset , we define the *Vaught transforms*

$$P^{\Delta U} = \{x \in X : \exists^* g \in U (g \cdot x \in P)\},$$

$$P^{*U} = \{x \in X : \forall^* g \in U (g \cdot x \in P)\},$$

where " $\exists^* g \in U$ " means "there exist nonmeager many $g \in U$ " and " $\forall^* g \in U$ " means "there exist comeager many $g \in U$ ".

An important point is that for X a Borel G -space and P Borel, $P^{\Delta U}$ and P^{*U} are Borel as well.

Lemma 1. Let G be a Polish group and Y a Polish G -space. If \mathcal{V} is a basis for Y and \mathcal{U}_1 a nbhd basis of $1 \in G$ the set

$$\mathcal{V}^{\Delta \mathcal{U}_1} = \{V^{\Delta N} : V \in \mathcal{V}, N \in \mathcal{U}_1\}$$

is a basis for Y .

Proof. It is easy to see that every set in $\mathcal{V}^{\Delta\mathcal{U}_1}$ is open. Let now $W \subseteq Y$ be open and $x \in W$. Let $x \in V \in \mathcal{V}$, $N \in \mathcal{U}_1$ be such that $N^{-1} \cdot V \subseteq W$. It is enough to show that $x \in V^N \subseteq W$. To use this notice that $\{g : g \cdot x \in V\}$ is an open nbhd of $1 \in G$, so it intersects N , thus $x \in V^{\Delta N}$. Now let $y \in V^{\Delta N}$, so that $\exists^* g \in N(g \cdot y \in V)$. Then for some $g \in N$, $g \cdot y \in V$, so $y \in g^{-1} \cdot V \subseteq N^{-1} \cdot V \subseteq W$. ⊢

Lemma 2 (Becker-Kechris [1996]). *Let G be a Polish group and X a Borel G -space. Let \mathcal{B} be a countable collection of Borel sets in X and \mathcal{U} a countable basis for G . Then there is a Polish topology $\tau_{\mathcal{B},\mathcal{U}}$ on X generating its Borel structure, such that the G -action on X is continuous and $\mathcal{B}^{\Delta\mathcal{U}} = \{B^{\Delta U} : B \in \mathcal{B}, U \in \mathcal{U}\} \subseteq \tau_{\mathcal{B},\mathcal{U}}$. If moreover, (X, τ) is given as a Polish G -space, then one can take $\tau_{\mathcal{B},\mathcal{U}} \supseteq \tau$.*

Let now $\mathcal{B} = f^{-1}(\mathcal{V}) = \{f^{-1}(V) : V \in \mathcal{V}\}$ and put $\tau_f = \tau_{\mathcal{B},\mathcal{U}}$. Since $f^{-1}(V^{\Delta N}) = [f^{-1}(V)]^{\Delta N}$, it follows that $f^{-1}(\mathcal{V}^{\Delta\mathcal{U}_1}) \subseteq \mathcal{B}^{\Delta\mathcal{U}} \subseteq \tau_{\mathcal{B},\mathcal{U}}$, where $\mathcal{U}_1 = \{N \in \mathcal{U} : 1 \in N\}$, and so by Lemma 1, $f : (X, \tau_f) \rightarrow Y$ is continuous and we are done. ⊢

We will now discuss two particular cases of this result.

Corollary (Becker-Kechris [1996]). *Let G be a Polish group. If X is a Borel G -space, there is a Polish topology τ on X , giving its Borel structure, so that (X, τ) is a Polish G -space. Equivalently, any Borel G -space is Borel isomorphic to a Polish G -space.*

The proof is obtained by applying the theorem to the trivial G -space $Y = \{y_0\}$, $g \cdot y_0 = y_0$, $f(x) = y_0 \in Y$. This corollary follows immediately from Kuratowski's Theorem for countable G , and was proved by Wagh [1988] for $G = \mathbb{R}$. It was raised as an open problem by D. Miller [1977] and, for locally compact G , by Ramsay [1985].

Corollary (Becker-Kechris [1996]). *Let G be a Polish group and X a Borel G -space. If $A \subseteq X$ is an invariant Borel set, there is a Polish topology τ_A , giving its Borel structure, so that (X, τ_A) is a Polish G -space and A is clopen in τ_A . If (X, τ) is given as a Polish G -space, then one can take $\tau_A \supseteq \tau$.*

Proof. Apply the theorem for $X, Y = \{0, 1\}$, the trivial action of G on Y , $g \cdot y = y$, and $f : X \rightarrow Y$ defined by $f(x) = 1$ iff $x \in A$. ⊢

This is again due to Kuratowski for G countable and to Sami [1994] for $G = S_\infty$. Some of Sami's ideas here have found their way in the proof of these more general results.

B An Application

Let us first mention one more corollary of the preceding theorem. (For the Fell topology on the space of closed subsets of a locally compact Polish space, see Kechris [1995].)

Corollary. *Let G be Polish locally compact, X a Borel G -space. Put on $\mathcal{F}(G)$ the Fell topology, so it is a compact Polish G -space under the conjugation action*

$$g \cdot F = gFg^{-1}.$$

Then there is a Polish topology τ on X , generating its Borel structure, so that (X, τ) is a Polish G -space and the map $x \mapsto G_x$ is continuous from (X, τ) into $\mathcal{F}(G)$.

Proof. Simply notice that $G_{g \cdot x} = gG_xg^{-1}$ and apply the theorem. \dashv

We will use this last corollary to provide a somewhat simplified and streamlined proof of the following descriptive strengthening of a measure theoretic result of Feldman-Hahn-Moore [1979]. This version of the proof was motivated by a communication of Ramsay.

Theorem. (Kechris [1992]). *Let G be a Polish locally compact group and X a Borel G -space. Then there exists a Borel lacunary complete section for X , i.e., a Borel set $S \subseteq X$ meeting every orbit of X , and such that for some open nbhd U of $1 \in G$ and all $s \in S$, $U \cdot s \cap S = \{s\}$. In particular, the intersection of S with every orbit is countable.*

Proof. By the preceding corollary, we can assume that X is a Polish G -space and $x \mapsto G_x$ is continuous. Fix a compatible metric d on X , a compact nbhd Λ of $1 \in G$ and a compact symmetric nbhd Δ of $1 \in G$ with $\Delta^2 \subseteq \Lambda$. Denote by Δ^0 the interior of Δ . Put

$$R_\Delta(x, y) \Leftrightarrow \exists g \in \Delta (g \cdot x = y).$$

Following Forrest [1974], let for $\varepsilon > 0$,

$$A_\varepsilon = \{x \in X : \forall g \in \Lambda (d(g \cdot x, x) \leq \varepsilon \Rightarrow g \in \Delta^0 G_x)\}.$$

Using the continuity of $x \mapsto G_x$ it is not hard to see that A_ε is open and then an argument by contradiction shows that $X = \bigcup_{n>0} A_{1/n}$. It is also easy to check that $R_\Delta|B$ is an equivalence relation, for any set $B \subseteq A_\varepsilon$ with $d(B) \leq \varepsilon$. So we can find a countable sequence of open subsets of X , $\{U_n\}$, such that R_Δ is a closed equivalence relation on each U_n . (It is closed, since Δ is compact.) So $R_n = R_\Delta|U_n$ is smooth, i.e., there is a Borel function $f_n : U_n \rightarrow [0, 1]$ with $xR_n y \Leftrightarrow f_n(x) = f_n(y)$ (see Kechris [1995]).

We next want to actually show that R_n has a Borel transversal, say T_n . If this is found, then let

$$S = \bigcup_n (T_n \setminus \bigcup_{m < n} [U_m]),$$

where $[U_m]$ = saturation of U_m , which easily works.

To prove the existence of T_n we use some descriptive set theory. Although in general it is not true that a closed equivalence relation has a Borel transversal, there are special circumstances under which this happens and these are satisfied here.

Definition. Let E be a Borel equivalence relation on the standard Borel space X . We say that E is *idealistic* if one can assign in a Borel way to each E -equivalence class C a σ -ideal I_C of subsets of C with $C \notin I_C$. (In a “Borel way” means that if $A \subseteq E$ is Borel, so is $\{x : A_x \in I_{[x]_E}\}$.)

Theorem (Kechris [1995]). *Let E be a smooth Borel equivalence relation on a standard Borel space X . If E is idealistic, then E has a Borel transversal.*

Finally, notice that each R_n is idealistic, since for each R_n -equivalence class C we can choose $x \in C$ and define:

$$B \in I_C \Leftrightarrow \{g : g \cdot x \in B\} \text{ is meager.}$$

Notice that this is independent of x , by the translation invariance of meagerness. Also $C \notin I_C$, since no open set in G is meager. \dashv

Lecture III

We will study from now on the orbit equivalence relation E_G of a group action and the orbit space $X/G = X/E_G$.

A Complexity of the Orbit Equivalence Relation

For G a Polish group and X a Borel G -space, let

$$xE_G^X y \Leftrightarrow xE_G y \Leftrightarrow \exists g(g \cdot x = y)$$

be the orbit equivalence relation. Recall that every orbit, i.e., every E_G -equivalence class, is Borel. However we have:

Theorem (Folklore). *The equivalence relation E_G is analytic but not in general Borel.*

Proof. Consider the logic action on structures of the form $\langle \mathbb{N}, R \rangle$, i.e., on the space $2^{\mathbb{N}^2}$. It can be shown that the corresponding E_G is not Borel (see, e.g., Kechris [1995]). \dashv

Under what circumstances is E_G actually Borel? Here are some well-known special cases (see Kechris [1995]).

Proposition. E_G is Borel if the action is free or if G is locally compact.

The following result characterizes when E_G is Borel.

Theorem (Becker-Kechris [1996]). *Let G be a Polish group and X a Borel G -space. Then the following are equivalent:*

- (i) E_G is Borel;
- (ii) $x \mapsto G_x$ (from X into $\mathcal{F}(G)$) is Borel.

Idea of the Proof. (ii) \Rightarrow (i) is classical. We sketch the proof that (i) \Rightarrow (ii), which is motivated by model theoretic ideas. For each $x \in X$, we fix a countable Boolean algebra \mathcal{B}_x of Borel sets, which depends “uniformly in a Borel way” (this is where the Borelness of E_G is used) on x , such that $G \cdot x \in \mathcal{B}_x$, $\mathcal{B}_x^{\Delta \mathcal{U}} \subseteq \mathcal{B}_x$ (with \mathcal{U} a countable basis for G), and the topology generated by \mathcal{B}_x is Polish. By Becker-Kechris [1996], the topology τ_x generated by $\mathcal{B}_x^{\Delta \mathcal{U}}$ is Polish and the action is continuous for (X, τ_x) . Since $G \cdot x = (G \cdot x)^{\Delta G} \in \tau_x$, by Effros’ Theorem the map $g \mapsto g \cdot x$ from G onto $G \cdot x$ is open (for τ_x restricted to $G \cdot x$). From this one can check the following formula: For $W, V \subseteq G$ open,

$$WG_x \cap V \neq \emptyset \Leftrightarrow \exists U \in \mathcal{U}, U \subseteq W \forall B \in \mathcal{B}_x (x \in B^{\Delta U} \Rightarrow x \in B^{\Delta V}),$$

which easily implies the Borelness of $x \mapsto G_x$. \dashv

Although in general E_G is not Borel, one has the following “approximation”.

Theorem (Becker-Kechris [1996]). *Let G be a Polish group and X a Borel G -space. There is a sequence $\{X_\alpha\}_{\alpha < \omega_1}$ of Borel sets such that $X = \bigcup_{\alpha < \omega_1} X_\alpha$ is a partition of X , each X_α is invariant, $E_G|_{X_\alpha}$ is Borel (in fact in a “uniform in α ” way) and moreover we have the following cofinality property: If $A \subseteq X$ is invariant Borel and $E_G|_A$ is Borel, then for some $\alpha < \omega_1$, $A \subseteq \bigcup_{\beta \leq \alpha} X_\beta$.*

The proof of this result uses the descriptive set theory of co-analytic sets.

B The Topological Vaught Conjecture

This is a basic question concerning the “effective” or “definable” cardinality of the orbit space X/G .

The Topological Vaught Conjecture (D. Miller, 1977). *Let G be a Polish group and X a Polish G -space. Either E_G has countably many classes or else there is a Cantor set $C \subseteq X$ such that $x, y \in C$, $x \neq y \Rightarrow \neg xE_G y$ (in which case we say that E_G has perfectly many classes).*

By $\text{TVC}(G)$ we abbreviate the statement that the above holds for the Polish group G , and we let

$$\text{TVC} \Leftrightarrow \forall G \text{ TVC}(G).$$

By the results in Lecture II, $\text{TVC}(G)$ is equivalent to its formulation for Borel G -spaces.

The TVC generalizes the famous *Vaught Conjecture* (VC) in model theory, which is the assertion that a first-order theory has either countably many or continuum many countable models (up to isomorphism). By Lecture I this is a special case of $\text{TVC}(S_\infty)$.

Both TVC and VC are open. We discuss below some progress that has been achieved to date.

First we remark that the analog of the TVC fails for analytic equivalence relations. However it holds for co-analytic equivalence relations. In particular it holds for Borel ones.

Theorem (Silver [1980]). *Let X be a Polish space and E a co-analytic equivalence relation on X . Then either E has countably many classes or else perfectly many classes.*

Idea of the Proof (due to Harrington). By a standard result of Mycielski, Kuratowski (see Kechris [1995]) an equivalence relation on a “reasonable” topological space S which is meager (in S^2) has perfectly many classes.

One now defines a new second countable “reasonable” topology τ on X , extending its given Polish topology, which is generated by a suitably chosen countable family of analytic sets. This is a version of the so-called Gandy-Harrington topology, which is defined using concepts of effective descriptive set theory. Then let $W = \bigcup_{C \in X/E} \text{Int}^\tau(C)$. If $X = W$, there are clearly only countably many classes of E . Otherwise, let $U = X \setminus W \neq \emptyset$. Magically it turns out that U is open (it is clearly closed) in τ and then one can check that $E|U$ is meager in (U^2, τ^2) , so it has perfectly many classes. \dashv

Corollary. *If E_G is Borel, for example if the action is free or G is locally compact, then E_G has countably many or perfectly many classes.*

Corollary. *$\text{TVC}(G)$ holds for Polish locally compact G .*

In another direction, but still making use of Silver’s Theorem, Sami [1994] proved $\text{TVC}(G)$ for every abelian Polish group G . This was extended by

Hjorth-Solecki [1999], who proved that $\text{TVC}(G)$ holds for all nilpotent Polish groups and all Polish groups admitting an invariant metric. Finally, very recently, Becker [1998] proved $\text{TVC}(G)$ for all Polish G which admit a complete left-invariant metric, and Hjorth [1997c] proved $\text{TVC}(G)$ for all Polish G with no closed subgroup which has S_∞ as a quotient, which is the widest class of groups known to satisfy the TVC to date.

C Glimm-Effros Dichotomies

A basic problem concerning a given G -space X is the “classification” of members of X up to orbit equivalence by “invariants”. This is a special case of the more general problem of classifying elements of a given standard Borel space X up to some equivalence relation E defined on that space.

Definition. Let E, E' be two equivalence relations on standard Borel spaces X, X' . We say that E is *Borel reducible* to E' , in symbols.

$$E \leq_B E'$$

if there is Borel $f : X \rightarrow X'$ such that

$$xEy \Leftrightarrow f(x)E'f(y).$$

Letting $\tilde{f}([x]_E) = [f(x)]_{E'}$ it is clear that $\tilde{f} : X/E \rightarrow X'/E'$ is an “embedding” of X/E into X'/E' .

Intuitively, $E \leq_B E'$ can be interpreted as meaning any one of the following:

- (i) E has a simpler classification problem than E' : any invariants for E' work for E as well (after composing with f).
- (ii) One can classify E -equivalence classes by invariants which take the form of E' -equivalence classes.
- (iii) The quotient space X/E “Borel embeds” into the quotient space X'/E' , so X/E has “definable cardinality” less than or equal to that of X'/E' .

Notation. If the function f above is actually 1-1 we put $E \sqsubseteq_B E'$. If it is moreover continuous, we let $E \sqsubseteq_c E'$.

In this notation, we can restate Silver’s Theorem as follows: Let for each set A , $\Delta(A)$ be the equality relation on A . Then for every co-analytic equivalence relation on a Polish space X , we have that exactly one of the following holds:

- (I) $E \leq_B \Delta(\mathbb{N})$;

(II) $\Delta(2^{\mathbb{N}}) \sqsubseteq_c E$.

Definition. Let E be a Borel equivalence relation on a standard Borel space X . We call E *concretely classifiable* or *smooth* if $E \leq_B \Delta(Y)$ for some standard Borel space Y (or equivalently if E has a countable Borel separating family $\{A_n\}$, i.e., $xEy \Leftrightarrow \forall n(x \in A_n \Leftrightarrow y \in A_n)$).

So if E is smooth, there is a Borel function $f : X \rightarrow Y$ (Y a standard Borel space) such that

$$xEy \Leftrightarrow f(x) = f(y).$$

Thus we can classify elements of X , up to E -equivalence, by invariants, computed in a Borel way, which are members of some standard Borel space.

Note the following equivalent formulation in the case of actions.

Theorem (Burgess [1979]; see also Kechris [1995]). *Let G be a Polish group and X a Borel G -space. Then if E_G is Borel, E_G is smooth iff E_G has a Borel transversal.*

This is a special case of the last theorem discussed in Lecture II.

Definition. E_0 is the following equivalence relation on $2^{\mathbb{N}}$:

$$xE_0y \Leftrightarrow \exists n \forall m \geq n (x(m) = y(m)).$$

This is (essentially) the equivalence relation induced by the odometer map and can be thought of as the combinatorial version of the classical Vitali equivalence relation on $[0, 1] : xE_Vy \Leftrightarrow \exists q \in \mathbb{Q}(q + x = y)$.

We now have

The Glimm-Effros Dichotomy (Effros [1965], [1981]). *Let G be a Polish group and X a Polish G -space for which E_G is F_σ . Then exactly one of the following holds:*

(I) E_G is smooth.

(II) $E_0 \sqsubseteq_c E_G$.

Alternatives (I) and (II) also have the following equivalents:

(I) (a) all the orbits are G_δ ;

(b) all the orbits are locally closed (i.e., the difference of two closed sets);

(II) There is an E_G -ergodic non-atomic probability measure on X .

One can derive the preceding result from the following theorem, which can be proved by a combinatorial construction.

Theorem (Becker-Kechris [1996]). *Let G be any group acting by homeomorphisms on a Polish space X . Assume there is a dense orbit and E_G is meager. Then $E_0 \subseteq_c E_G$.*

To see how to prove the Glimm-Effros Dichotomy from this, consider the generic ergodic decomposition of X , i.e., define the following equivalence relation:

$$x \tilde{E}_G y \Leftrightarrow \overline{[x]_{E_G}} = \overline{[y]_{E_G}}.$$

It is easy to see that $\tilde{E}_G (\supseteq E_G)$ is a G_δ equivalence relation, whose equivalence classes are (of course) G_δ sets on which the G -action is minimal (i.e., all orbits are dense). If $E_G = \tilde{E}_G$, then every orbit is G_δ , so the map $x \mapsto \overline{[x]_{E_G}}$ (from X into the standard Borel space $\mathcal{F}(X)$ of all closed subsets of X with the Effros Borel structure) shows that E_G is smooth. Otherwise, one \tilde{E}_G -equivalence class, say C , contains at least two orbits and since every orbit is F_σ and dense in C , it follows, from the Baire Category Theorem, that every orbit in C is meager, so by the Kuratowski-Ulam Theorem, $E_G|C$ is meager (in C^2), so $E_0 \subseteq_c E_G|C$ and thus $E_0 \subseteq_c E_G$.

In 1990 the Glimm-Effros Dichotomy has been extended to the general context of Borel equivalence relations.

Theorem (Harrington-Kechris-Louveau [1990]). *Let E be a Borel equivalence relation on a Polish space X . Then exactly one of the following holds:*

- (I) E is smooth;
- (II) $E_0 \subseteq_c E$.

Moreover, (I) is equivalent to the existence of a Polish topology σ on X , extending its given topology, so that E is closed in σ , and (II) is equivalent to the existence of an E -ergodic, non-atomic probability Borel measure on X .

The proof uses the “change of topology” idea. One defines, as in the proof of Silver’s Theorem, a new topology τ , using effective descriptive set theory, and then considers E in (X^2, τ^2) : If E is closed in (X^2, τ^2) , then it turns out that E is smooth, while otherwise one can show that $E_0 \subseteq_c E$.

It follows that one has a full Glimm-Effros Dichotomy for E_G , provided that it is Borel. However, any reasonable form of Glimm-Effros Dichotomy for general Borel or Polish G -spaces fails, if in the first alternative we require the classifying invariants to be members of a standard Borel space (even for $G = S_\infty$). However, we have the following version of this dichotomy for

arbitrary Borel G -spaces, in which the classifying invariants are now replaced by countable transfinite sequences of 0's and 1's. This is motivated by a classical result of Ulm on classifying countable abelian p -groups (see Fuchs [1970]).

Theorem (Becker, Hjorth-Kechris [1995]). *Let G be a Polish group and X a Polish G -space. Then exactly one of the following holds:*

- (I) E_G can be classified (definably) by invariants which are countable transfinite sequences of 0's and 1's (Ulm-type classification);
- (II) $E_0 \subseteq_c E_G$.

In another direction, one can recover the original Glimm-Effros Dichotomy, for arbitrary Polish G -spaces, by considering restricted classes of groups G , which are somehow nicer.

To motivate the next definition, notice that if X is a Polish G -space with E_G in F_σ and the action is minimal, then if one orbit is non-meager, the action is actually transitive. This, by using the generic ergodic decomposition, implies the Glimm-Effros Dichotomy in the following strong form: either every orbit is G_δ or $E_0 \subseteq_c E$.

Definition. We say that a Polish group G is a GE -group if every minimal Polish G -space with a non-meager orbit is transitive.

Definition. We say that the Polish group G satisfies the *strong Glimm-Effros Dichotomy* if for every Polish G -space X one of the following holds: every orbit is G_δ or $E_0 \subseteq_c E_G$.

Thus every GE -group satisfies the strong Glimm-Effros Dichotomy. The following are known to be GE -groups:

- (i) locally compact groups (since then E_G is F_σ);
- (ii) (Hjorth-Solecki [1999]) nilpotent or having an invariant metric groups;
- (iii) (Hjorth [1996]) countable products of locally compact Polish groups.

Solecki has shown that every GE -group admits a complete left-invariant metric. However Hjorth-Solecki [1999] have found examples of solvable Polish groups (of rank 2), which fail to satisfy the strong Glimm-Effros Dichotomy. On the other hand, Hjorth-Solecki [1999] and Kechris showed that it holds even for Polish groups admitting a complete left-invariant metric for all *free* actions.

Finally, very recently, Becker showed that although Polish groups which admit complete left-invariant metrics fail to satisfy the strong Glimm-Effros

Dichotomy, they still satisfy the Glimm-Effros Dichotomy, in fact, in the following form.

Theorem (Becker [1998]). *Let G be a Polish group admitting a complete left-invariant metric. Let X be a Polish G -space. Then either every orbit is Π^0_ω or else $E_0 \sqsubseteq_c E$. Since in the first alternative E_G is actually Borel, we have that either E_G is smooth or else $E_0 \sqsubseteq_c E_G$, so the Glimm-Effros Dichotomy holds for such G .*

Lecture IV

Turbulence

We will describe here very recent work of G. Hjorth [1997a] (see also Kechris [1997] for an exposition of Hjorth's results).

Definition. Let E be an equivalence relation on a standard Borel space X . We say that E *admits classification by countable structures* if there is a Borel map assigning to each $x \in X$ a countable structure \mathcal{A}_x (with domain \mathbb{N} in some countable language) such that

$$xEy \Leftrightarrow \mathcal{A}_x \cong \mathcal{A}_y,$$

i.e., invariants are countable structures up to isomorphism.

It is easy to see that if E is smooth, then it admits classification by countable structures (but the converse easily fails). Also by Lecture I, E admits classification by countable structures iff $E \leq_B E_{S_\infty}^Y$ for some Borel S_∞ -space Y .

Examples.

1. Let D be the set of all ergodic $T \in \text{Aut}(X, \mu)$ which have discrete spectrum. Let E_C be conjugacy in $\text{Aut}(X, \mu)$ restricted to D . (So E_C is induced by the conjugacy action of $\text{Aut}(X, \mu)$ on D .) By the Halmos-von Neumann Theorem, for $S, T \in D$:

$$\begin{aligned} SE_C T &\Leftrightarrow \{\lambda \in \mathbb{T} : \lambda \text{ is an eigenvalue of } S\} \\ &= \{\lambda \in \mathbb{T} : \lambda \text{ is an eigenvalue of } T\}. \end{aligned}$$

Restricting ourselves for convenience to the set of $T \in D$ with infinite spectrum (otherwise we have to modify appropriately what follows), we can find a Borel function $f : D \rightarrow \mathbb{T}^{\mathbb{N}}$ such that $f(T) = \{x_n\}$, where $\{x_n\}$ is a 1-1 enumeration of the set of eigenvalues of T . Let S_∞ act on $\mathbb{T}^{\mathbb{N}}$ by $g \cdot \{x_n\} = \{x_{g(n)}\}$. Then

$$SECT \Leftrightarrow f(x)E_{S_\infty}f(y),$$

i.e., $E_C \leq_B E_{S_\infty}$ (for this action), so E_C admits classification by countable structures.

2. (This comes from recent work of Giordano-Putnam-Skau [1995]). Consider minimal $f \in H(2^{\mathbb{N}})$. Let

$$fEg \Leftrightarrow \exists h \in H(2^{\mathbb{N}})(h \text{ maps the} \\ \text{orbits of } f \text{ onto the orbits of } g).$$

Then Giordano-Putnam-Skau show (among other things) that one can assign to each $f \in H(2^{\mathbb{N}})$ a countable partially ordered group with distinguished order unit, \mathcal{A}_f , such that

$$fEg \Leftrightarrow \mathcal{A}_f \cong \mathcal{A}_g;$$

so E admits classification by countable structures.

We now consider the following general question: Given a Polish G -space X , when does E_G admit classification by countable structures?

Definition. Let G be a Polish group and X a Polish G -space. Fix an open nonempty set $U \subseteq X$ and a symmetric open nbhd V of $1 \in G$. The (U, V) -local graph is the following symmetric, reflexive relation on U :

$$xR_{U,V}y \Leftrightarrow x, y \in U \text{ and } \exists g \in V(g \cdot x = y).$$

The (U, V) -local orbit of $x \in U$, $\mathcal{O}(x, U, V)$, is the connected component of x in this graph.

Definition. The Polish G -space X is *turbulent* if every orbit is dense and meager, and every local orbit is somewhere dense (i.e., its closure has non-empty interior).

Examples. Let $\mathbb{R}^{<\mathbb{N}} \subseteq G \subsetneq \mathbb{R}^{\mathbb{N}}$ be a Polishable subgroup, i.e., a Borel subgroup of $\mathbb{R}^{\mathbb{N}}$ which is Borel isomorphic to a Polish group. An example of such a G is ℓ^p , for $1 \leq p < \infty$. Then the translation action of G on $\mathbb{R}^{\mathbb{N}}$ is turbulent (when G is viewed as a Polish group). Similarly for many Polishable subgroups of $\mathbb{Z}_2^{\mathbb{N}}$. On the other hand Hjorth and Kechris have shown that any closed subgroup of a countable product of locally compact groups and closed subgroups of S_∞ never has turbulent actions.

We now have:

Theorem (Hjorth [1997a]; see also Kechris [1996]). *Let G be a Polish group and X a turbulent Polish G -space. Then E_G^X does not admit classification by countable structures.*

In fact, if we call a Polish G -space *generically turbulent*, if its restriction to an invariant dense G_δ is turbulent, we have the following characterization.

Theorem (Hjorth [1997a]). *Let G be a Polish group and X be a Polish G -space, such that X has a dense orbit and every orbit is meager. Then the following are equivalent:*

- (i) X is generically turbulent;
- (ii) If Y is a Borel S_∞ -space and $f : X \rightarrow Y$ is Baire measurable, such that $xE_Gy \Rightarrow f(x)E_{S_\infty}F(y)$, then F maps a comeager set into a single E_{S_∞} -class, i.e., E_G is generically E_{S_∞} -ergodic.

One now has the following dichotomy, at least for GE -groups.

Theorem (Hjorth [1997a]). *Let G be a GE -group. Then exactly one of the following happens for each Polish G -space X :*

- (I) E_G^X admits classification by countable structures;
- (II) There is a turbulent Polish G -space Y with $E_G^Y \leq_B E_G^X$.

This is proved by an appropriate “changing the topology” technique. Using model theoretic ideas – an analog of the Scott analysis of countable structures – one assigns in a Borel way to each $x \in X$ a countable structure \mathcal{A}_x (a partial ordering with some additional relations) and an $L_{\omega_1\omega}$ sentence σ_x which is a weak version of a Scott sentence of \mathcal{A}_x , such that $xE_G^Xy \Rightarrow \mathcal{A}_x \cong \mathcal{A}_y \Rightarrow \sigma_x = \sigma_y$. If $xE_G^Xy \Leftrightarrow \sigma_x = \sigma_y$, then clearly E_G^X admits classification by countable structures, so we have alternative (I). Otherwise, one can define a new Polish topology on an invariant Borel set $Y \subseteq X$, containing x, y with $\sigma_x = \sigma_y$ but $x \not E_G^X y$, so that the action of G on Y is continuous and turbulent, thus we have alternative (II).

The following application of these results has been observed by Kechris: Measure equivalence and conjugacy on $U(H)$ do not admit classification by countable structures (in contrast with the examples above concerning automorphism with countable discrete spectrum). It was conjectured that conjugacy on $\text{Aut}(X, \mu)$ does not admit classification by countable structures. This has now been proved by Hjorth [1997b]. At the time of the workshop it was also open whether conjugacy on $U(H)$, $\text{Aut}(X, \mu)$ is generically turbulent. It has been recently proved by Kechris-Sofronidis [1997] that indeed conjugacy on $U(H)$ is generically turbulent but this is still open for $\text{Aut}(X, \mu)$.

A somewhat weaker version of the preceding dichotomy has been also proved by Hjorth for arbitrary Polish groups G . Write, for equivalence relations E, F on X, Y resp.,

$$E \leq_{p\Delta_2^1} F$$

iff there is a provably Δ_2^1 function $f: X \rightarrow Y$ with $xEy \Leftrightarrow f(x)Ff(y)$.

Theorem (Hjorth [1997a]). *Let G be a Polish group. Then exactly one of the following holds for each Polish G -space X :*

- (I) $E_G^X \leq_{p\Delta_2^1} E_{S_\infty}^Z$, for some Borel S_∞ -space Z ;
- (II) *There is a turbulent Polish G -space Y with $E_G^Y \leq_B E_G^X$.*

(I) in the preceding theorem essentially says that E_G^X admits classification by countable structures, albeit in a somewhat weaker form, since the classifying map is $p\Delta_2^1$ but not necessarily Borel. Thus, intuitively speaking, this shows that even for arbitrary Polish groups G , the precise obstruction for classifying orbit equivalence relations E_G by countable structures is turbulence. It would be nice to replace " $p\Delta_2^1$ " by "Borel", so that one has the full Hjorth dichotomy valid for arbitrary (not just GE groups), but this is still open.

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